Применение определенного интеграла.

Площадь плоской фигуры.

$$M[1] = \iint\limits_D 1 dxdy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} 1 dy\,,$$

$$M[1] = \int_a^b \left(y_2(x) - y_1(x)\right) dx\,.$$

$$M[x] = \iint\limits_D x dxdy \ = \int_a^b x dx \int_{y_1(x)}^{y_2(x)} dy\,,$$

$$M[x] = \int_a^b x \left(y_2(x) - y_1(x)\right) dx\,.$$

$$\langle x \rangle = \frac{M[x]}{M[1]}.$$

$$M[x^2] = \iint\limits_D x^2 dxdy = \int_a^b x^2 dx \int_{y_1(x)}^{y_2(x)} dy \,,$$

$$M[x^2] = \int_a^b x^2 \left(y_2(x) - y_1(x) \right) dx \,.$$

$$\left\langle x^2 \right\rangle = \frac{M[x^2]}{M[1]},\; D = \left\langle x^2 \right\rangle - \langle x \rangle^2 .$$

$$M[y] = \iint\limits_D ydxdy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} ydy \,,$$

$$M[y] = \int_a^b dx \left. \frac{y^2}{2} \right|_{y_1}^{y_2} = \int_a^b \frac{y_2^2(x) - y_1^2(x)}{2} dx \,.$$

$$M[y] = \int_a^b (y_2(x) - y_1(x))\frac{y_2(x) + y_1(x)}{2}\,dx\,.$$

$$\langle y \rangle = \frac{M[y]}{M[1]}\,.$$

Объем тела вращения относительно оси OX

$$VOX[1] = \iint\limits_D 2\pi y dx dy = 2\pi \int_a^b dx \int_{y_1(x)}^{y_2(x)} y dy\,,$$

$$VOX[1] = \pi \int_a^b \left(y_2^2(x) - y_1^2(x)\right) dx\,.$$

$$VOX[x] = \iint\limits_D x 2\pi y dx dy\,,$$

$$VOX[x] = \pi\int_a^b x\left(y_2^2(x) - y_1^2(x)\right)dx\,.$$

$$VOX.\langle x\rangle = \frac{VOX[x]}{VOX[1]}.$$

Объем тела вращения относительно оси OY

$$VOY[1] = \iint_D 2\pi x dx dy = 2\pi\int_a^b x dx\int_{y_1(x)}^{y_2(x)} dy\,,$$

$$VOY[1] = 2\pi\int_a^b x\left(y_2(x) - y_1(x)\right)dx\,.$$

$$VOY[y] = \iint_D 2\pi x y dx dy\,,$$

$$VOY[y] = \pi \int_a^b x \left(y_2^2(x) - y_1^2(x) \right) dx \,.$$

$$VOY.\langle y \rangle = \frac{VOY[y]}{VOY[1]}.$$

Пример:

$$f(x) = x^5,\ x \in [0;1],\ 0 \le y \le f(x).$$

$$M[1] = \int_0^1 \left(x^5 - 0\right) dx = \left.\frac{x^6}{6}\right|_0^1 = \frac{1}{6}.$$

$$M[x] = \int_0^1 x\left(x^5 - 0\right) dx = \left.\frac{x^7}{7}\right|_0^1 = \frac{1}{7}.$$

$$\langle x \rangle = \frac{1/7}{1/6} = \frac{6}{7}.$$

$$M[x^2] = \int_0^1 x^2\left(x^5 - 0\right) dx = \left.\frac{x^8}{8}\right|_0^1 = \frac{1}{8}.$$

$$\left\langle x^2 \right\rangle = \frac{1/8}{1/6} = \frac{3}{4}.$$

$$D = \left\langle x^2 \right\rangle - \langle x \rangle^2$$

$$D = \frac{3}{4} - \left(\frac{6}{7}\right)^2 = \frac{3}{4} - \frac{36}{49} = \frac{147-144}{196} = \frac{3}{196}.$$

$$M[y] = \int_0^1 \frac{x^{10}}{2} dx = \left.\frac{x^{11}}{22}\right|_0^1 = \frac{1}{22}.$$

$$\langle y \rangle = \frac{1/22}{1/6} = \frac{6}{22} = \frac{3}{11}.$$

Объем тела вращения относительно оси OX

$$VOX[1] = \pi \int_0^1 x^{10} dx = \pi \left. \frac{x^{11}}{11} \right|_0^1 = \frac{\pi}{11}.$$

$$VOX[x] = \pi \int_0^1 x^{11} dx = \pi \left. \frac{x^{12}}{12} \right|_0^1 = \frac{\pi}{12}.$$

$$VOX\langle x \rangle = \frac{\pi / 12}{\pi / 11} = \frac{11}{12}.$$

Объем тела вращения относительно оси OY

$$VOY[1] = 2\pi \int_0^1 x^6 dx = 2\pi \left. \frac{x^7}{7} \right|_0^1 = \frac{2\pi}{7}.$$

$$VOY[y] = \pi\int_0^1 x^{11}dx = \pi\,\frac{x^{12}}{12}\bigg|_0^1 = \frac{\pi}{12}.$$

$$VOY.\langle y\rangle = \frac{\pi\,/\,12}{2\pi\,/\,7} = \frac{7}{24}.$$

Пример:

$$f(x) = \sin x\,,\ x\in[0;\pi].$$

$$M[1] = \iint\limits_D 1dxdy = \int_0^{\pi}\sin x\,dx\,,$$

$$M[1] = -\cos x|_0^{\pi} = 2,$$

$$M[x] = \int_0^\pi x \sin x \, dx = -\int_0^\pi x d \cos x$$

$$= -x \cos x|_0^\pi + \int_0^\pi \cos x \, dx = \pi \, .$$

$$\langle x \rangle = \frac{\pi}{2}.$$

$$VOX[1] = \pi \int_0^\pi \sin^2 x dx = \frac{\pi^2}{2}.$$

$$VOX[x] = \pi \int_0^\pi x \sin^2 x dx = \frac{\pi^3}{4}.$$

$$VOX.\langle x \rangle = \frac{\pi^3 / 4}{\pi^2 / 2} = \frac{\pi}{2}.$$

$$VOY[1] = 2\pi\int_0^\pi x\sin x dx = 2\pi^2.$$

$$VOX[y] = \pi\int_0^\pi x\sin^2 x dx = \frac{\pi^3}{4}.$$

$$VOY.\langle y\rangle = \frac{\pi^3 / 4}{2\pi^2} = \frac{\pi}{8}.$$